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The W sequence for circle maps and misbehaved itineraries

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Abstract. The W sequence, an ordering of maximal itineraries for circle maps, is constructed by means of the Farey transformation and the gluing operation. The similarity in the dynamical behaviour of the maps is explicitly described in terms of iterinaries. A class of badly ordered itineraries, termed misbehaved, is defined with the language of symbolic sequences. A misbehaved itinerary is shown to imply a rotational interval.

Maps of a circle to itself have been widely used to model many systems exhibiting transition from an orientation-preserving motion to chaos [1]. The transition has been studied by many authors. Recent results on all the routes can be found in [2], which also includes a coherent account of many well known results. In this paper we shall describe circle maps using the language of symbolic sequences.

We restrict attention to a class of maps which are increasing except perhaps for one inteval M where they are decreasing. We may divide the circle into three parts: the left increasing region L, the right increasing region R and the leftmost decreasing region M according to the monotonicity of the given mapping function (figure 1). We mark any point by the letter R, L or M according to the region in which it falls. The subsequent iterations of an initial point then generates a sequence of the symbols R, L and M, the so-called itinerary. This coarse-grained description of the dynamics reflects the essential feature of the evolution process. We assign an order among the

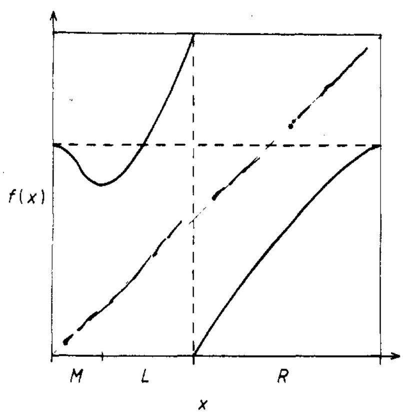

Figure 1. The circle is divided into three parts: the left increasing region L, the right increasing region R and the leftmost decreasing region M.

symbols R, L and M according to their natural order, i.e. $R > L > M$. For two itineraries with the first n symbols identical we define their order as the order of the $(n+1)$th symbol if the number of the symbol M in those first n symbols is even, otherwise we reverse the order. Let us first consider only itineraries containing no letter M. The itineraries which never have shifts greater than themselves are termed maximal. In this paper we only consider itineraries which are maximal. The ordering of certain maximal itineraries according to the order defined above is called the W sequence, where W stands for winding number or simply for word. Its counterpart for the unimodal maps on the interval is the *MSS* sequence [3].

Before constructing the W sequence let us summarise some preliminary knowledge about the winding number and the Farey tree. We can define the winding number of a point or its itinerary (containing no M) as the limit of the ratio of R to the total number of symbols in longer and longer truncation of the itinerary [4]. This definition is a symbolic version of the one usually adopted. For a given map the set of the possible winding numbers always forms a point or a non-empty closed interval the rotational interval [2, 5]. Rational winding numbers can conveniently be arranged on the Farey tree [6] (figure 2). A rational number $r/s = [a_0, a_1, \ldots, a_n]$, $a_n > 1$, called a daughter, has her father $p/q = [a_0, a_1, \ldots, a_{n-1}]$ and mother $m/n = [a_0, a_1, \ldots, a_n - 1]$. The daughter is the Farey sum of her parents, i.e.

$$\frac{r}{s} = \frac{m}{n} \qquad \text{and} \qquad \frac{p}{q} = \frac{m+p}{n+q}, \tag{1}$$

For a given mother $m/n = [b_0, b_1, \ldots, b_n]$, with $b_n > 1$, there are always two daughters $[b_0, b_1, \ldots, b_n + 1]$ and $[b_0, b_1, \ldots, b_n - 1, 2]$. Bonds are drawn between the mother and her two daughters, and the left bond, labelled -1, is connected to the smaller of the two, which has an odd number of entries in its continued fraction expansion. The right bond, labelled $+1$, is similarly connected to the greater daughter with an even number of entries. The binary tree rooted at the top element 1/2 is the Farey tree. The extended tree includes 0/1 and 1/1 as level 0. The Farey address of an element in the tree is defined to be the string of ± 1 obtained by reading the path from the top of the tree to the element, e.g. $2/7 = \langle \bar{1}\bar{1}1 \rangle$ (see figure 2). The Farey address of an

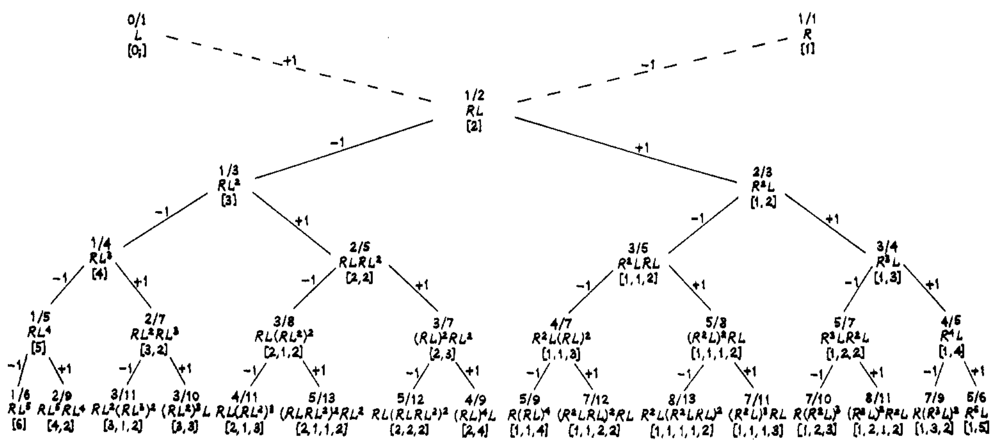

Figure 2. The extended Farey tree through level five.

element on the tree can easily be obtained from its continued fraction expansion and the relation between a daughter and her mother.

In figure 2 a string Σ of the letters R and L is shown together with a fraction number and its continued fraction expansion. The maximal itinerary Σ^∞ has a winding number equal to the corresponding fraction number. If the two parents have itineraries Σ^∞ and Δ^∞, and $\Sigma > \Delta$, then their daughter has the itinerary $(\Sigma\Delta)^\infty$, formed by just gluing the itineraries of the parents together in a proper order, and $\Sigma^\infty > (\Sigma\Delta)^\infty > \Delta^\infty$. In this way we can generate the tree of itineraries and transfer the order of fraction numbers to itineraries. It can be seen that this procedure preserves the maximality of itineraries. There is another way to generate an itinerary corresponding to any fraction number by using its Farey address. To do this, let us define the Farey transformation for the symbols [6]

$$\mathcal{T}_{+1}: R \to R,\ L \to RL \tag{2a}$$

and

$$\mathcal{T}_{-1}: R \to RL,\ L \to L. \tag{2b}$$

The transformation of the given string $\Sigma = s_0 s_1 \ldots s_n$ of the letters R and L is then defined as

$$\mathcal{T}(\Sigma) = \mathcal{T}_\varepsilon(s_0)\mathcal{T}_\varepsilon(s_1)\ldots\mathcal{T}_\varepsilon(s_n) \qquad \varepsilon = \pm 1. \tag{3}$$

The relation between the winding number of the transformed itinerary and that of the original one can be easily derived. It can be proved that for an element with the Farey address $\langle b_1, b_2 \ldots b_n\rangle$ the corresponding string is

$$\mathcal{T}_{b_1}(\mathcal{T}_{b_2}(\ldots(\mathcal{T}_{b_n}(RL))\ldots)). \tag{4}$$

For example, we construct the string of $5/13 = \langle \bar{1}1\bar{1}1\rangle$ as follows:

$$RL \xrightarrow{+1} RRL \xrightarrow{-1} (RL)^2 L \xrightarrow{+1} (RRL)^2 RL \xrightarrow{-1} (RLRL^2)^2 RL^2$$

as shown in figure 2. The Farey transformation has the remarkable property of preserving the maximality of itineraries.

We now construct the W sequence. We regard the whole sequence as a dictionary which consists of infinitely many volumes, one for each winding number. Let us adopt the convention of indicating the periodic itinerary Σ^∞ by its fundamental string Σ of the shortest length. We then describe the contents of a volume for a given winding number. We call volume rational when the corresponding winding number is rational, otherwise it is irrational. The simplest rational volume belongs to the winding number 1/2. We 'print' the word RL on the title page. The last word of the volume is $R(RL)^\infty$. Any word inside the volume is of the form

$$R(RL)^{m_1}(LR)^{n_1}\ldots(RL)^{m_k}(LR)^{n_k}L. \tag{5}$$

To guarantee the maximality of an itinerary, certain conditions such as $m_1 > m_i$, or $m_1 = m_i$ but $n_1 < n_i$ for $i > 1$ should be satisfied. Inside the volume words are arranged according to their natural order, e.g. the power m_1 is non-decreasing in the volume. The word series of the volume is the counterpart of the word series from $(RR)^\infty$ to $RL(RR)^\infty$ in the unimodal map on the interval. Any other rational volume is similar to the volume of RL with the letters R and L replaced by strings ρ and λ. If the winding number of the volume written in the Farey address is $\langle b_1 b_2 \ldots b_n\rangle$, then we have

$$\rho = \mathcal{T}_{b_1}(\mathcal{T}_{b_2}(\ldots(\mathcal{T}_{b_n}(R))\ldots)), \qquad \lambda = \mathcal{T}_{b_1}(\mathcal{T}_{b_2}(\ldots(\mathcal{T}_{b_n}(L))\ldots)). \tag{6}$$

An irrational number has an infinitely long Farey address. Its volume consists of a single word and is constructed as follows. Write down the word segment RL first. Passing from the top along the bonds towards the irrational number, at every daughter turning point copy the title page of its father as a word segment for the volume and put the page on the top or bottom of the pages already obtained according to whether the father is larger or smaller than the daughter.

So far we have mentioned nothing about the letter M. Obviously, $RL > RM > L$. It can then be seen that if both itineraries $\Sigma RL\Delta$ and $\Sigma L\Delta$ are maximal, then $\Sigma RM\Delta$ must exist and be maximal as well, and moreover must be ordered between the two itineraries. Denote by $\omega(\Sigma)$ the winding number of the itinerary if it exists. It is then generally true that $\omega(\Sigma RM\Delta)$ is either equal to $\omega(\Sigma RL\Delta)$ or to $\omega(\Sigma L\Delta)$.

Finally, the W sequence is constructed as the collection of volumes arranged in order of increasing winding number.

We now describe the similarity in the structure of the W sequence. The similarity is twofold: inter-voluminous and intra-voluminous. For the inter-voluminous similarity nothing need be said since it is clearly seen from the way in which a volume other than the RL volume of the winding number 1/2 is constructed. The intra-voluminous similarity can be seen, for example, by looking at the RL volume. This volume contains $(RL)^\infty = R(LR)^\infty$ as the first word and $R(RL)^\infty$ as the last word. By dropping the first letter R of every word (in the infinite form) or by cycling letters of every word to the left by one letter and renaming RL and LR as ρ and λ, the volume includes a pseudo-copy of the whole dictionary. It is then easy to see that by cutting the part ranging from the word $R(\rho\lambda)^\infty = R(RLLR)^\infty$ to $R\rho(\rho\lambda)^\infty = RRL(RLLR)^\infty$ out of the volume, a pseudo-copy of the volume itself is obtained. We also have the inter-voluminous similarity in this sub-voluminous level. Furthermore, we can manipulate the procedure *ad infinitum.*

In the above we have not discussed badly ordered itineraries [7] in detail. All the words inside any volume except for the title and the last one are badly ordered. Any itinerary obtained by gluing together two itineraries which are not the parent pair of a daughter or the Farey neighbours is badly ordered. A badly ordered itinerary can be generated in this way, and may perhaps involve more than two ordered itineraries. For two given non-neighbouring itinerary elements on the tree, Σ and Δ, there is a coarse-grained chaotic set in between, members of which are of the form

$$\Sigma^{m_1}\Delta^{n_1}\Sigma^{m_2}\Delta^{n_2}\dots .$$

The chaotic set, corresponding to a rotational interval, is responsible for the chaotic behaviour. In this case it is easy to construct itineraries without a winding number [5].

In constructing any real itinerary Σ, the following admissibility condition should be satisfied:

$$\mathcal{S}_L(\Sigma) \geq K_- \qquad \mathcal{S}_R(\Sigma) \leq K_+ \tag{7}$$

where $\mathcal{S}_L(\Sigma)$ indicates shifted itineraries of Σ that immediately follow any letter L in Σ, $\mathcal{S}_R$ is analogous, and K_- and K_+ are the itineraries of the two critical points connecting with the branches L and R, respectively. (The condition for $\mathcal{S}_M$ is: $K_- \leq \mathcal{S}_M(\Sigma) \leq K_+$.) For given $K_\pm$ any itinerary of R and L satisfying the condition (7) is admissible, and so corresponds to a certain point on the circle.

The language of symbolic sequences provides a tool for a fine description of badly ordered itineraries. A badly ordered periodic itinerary exhibits a definite winding number, say 1/2. Since $\omega = 1/2$ is a winding number for the map, there must exist the ordered itinerary $(RL)^\infty$ [8]. This can be easily seen from symbolic dynamics. Any badly ordered itinerary Θ can be generated from the word $(RL)^\infty$ by replacing RL with some other segments such as LR. Thus, for any badly ordered itinerary of the winding number 1/2 we must have

$$K_+ \geq R(RL)\ldots \qquad \text{and} \qquad K_- \leq L(LR)\ldots. \tag{8}$$

Of course, for such $K_\pm$ the ordered itinerary $(RL)^\infty$ is admissible. More generally, an admissible itinerary that contains the segment $R(RL)^m(LR)^nR$, with m and n being some integers, implies

$$K_+ \geq R(RL)^m(LR)^nR\ldots \qquad \text{and} \qquad K_- \leq L(LR)^mR\ldots.$$

Consequently, such an itinerary guarantees that itineraries

$$[R(RL)^{m'}(LR)^{n'}L]^\infty \qquad \text{with } m' < m \text{ and } n' < n-1$$

are also admissible.

Any badly ordered itinerary of the winding number 1/2 containing the segment $R(RL)^mRR$ or $L(LR)^nLL$ will be termed *misbehaved*. When $R(RL)^mRR$ is contained we can determine K_+, more precisely than in (8), to be

$$K_+ \geq R(RL)^mRR\ldots$$

which, combined with the condition $K_- \leq L(LR)\ldots$, guarantees the admissibility of the ordered itinerary $[R(RL)^{m+1}]^\infty$ of the winding number $\omega_+ = (m+2)/(2m+3)$. Thus, there must exist a rotational interval including the interval $[\frac{1}{2}, \omega_+]$. Similarly, for an itinerary of the winding number 1/2 containing the segment $L(LR)^nLL$ we have $K_- \leq L(LR)^nLL\ldots$. We can then verify the admissibility of the ordered itinerary $[L(LR)^{n+1}]^\infty$, hence the existence of a rotational interval including $[(n+1)/(2n+3), \frac{1}{2}]$.

We now discuss badly ordered itineraries of a winding number ω other than 1/2. The ordered itinerary of the winding number ω can be obtained from $(RL)^\infty$ through the Farey transformations (see equation (4)). Let us denote it by $(\mathscr{R}\mathscr{L})^\infty$. It can be seen that $\mathscr{R}^\infty$ and $\mathscr{L}^\infty$ correspond to the parents of ω. Denote by ω_R and ω_L their winding numbers, so $\omega_R > \omega > \omega_L$. We define misbehaved itineraries of the winding number ω to be those which contain a segment greater than $\mathscr{R}(\mathscr{R}\mathscr{L})^m$ or smaller than $\mathscr{L}(\mathscr{L}\mathscr{R})^n$. It is obvious that any badly ordered string can be decomposed into several ordered segments. If a misbehaved itinerary Θ contains an ordered segment $\mathscr{R}_+$ greater than $\mathscr{R}$, we can deduce that $K_+ \geq \mathscr{S}_R(\mathscr{R}_+\ldots) \geq \mathscr{S}_R(\mathscr{R}^\infty)$ and $\mathscr{S}_L(\mathscr{R}^\infty) \leq \mathscr{S}_L(\mathscr{R}_+\ldots) \leq K_-$. $\mathscr{R}^\infty$ is then admissible for the map. Similarly, when Θ contains an ordered segment $\mathscr{L}_-$ smaller than $\mathscr{L}$, $\mathscr{L}^\infty$ is admissible. Thus, there exists a rotational interval including $[\omega, \omega_R]$ or $[\omega_L, \omega]$). When neither $\mathscr{R}_+$ nor $\mathscr{L}_-$ appears, Θ involves only the segments $\mathscr{R}$ and $\mathscr{L}$ since any ordered segments between the segments $\mathscr{R}$ and $\mathscr{L}$ consists of $\mathscr{R}$ and $\mathscr{L}$ only. As one can see from the self-similarity, all the arguments in the proof of a rotational interval for a misbehaved itinerary with respect to the RL volume can now be directly applied to the proof for a misbehaved itinerary with respect to the $\mathscr{R}\mathscr{L}$ volume. We can then conclude that a misbehaved itinerary implies a rotational interval.

A word inside the $\mathscr{R}\mathscr{L}$ volume may be misbehaved in a coarse-grained sense, although originally it is not misbehaved.

In the above we have proved by means of symbolic dynamics that for any badly ordered periodic itinerary there exists an ordered itinerary of the same winding number for the circle map. Furthermore, we have found a class of badly ordered itineraries with respect to a given winding number which imply the existence of a rotational interval.

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